Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	12	12	100	98	65
				Total	100

Instructions to candidates

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2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

SN042-205-3

Section One: Calculator-free **35% (52 Marks)**

This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(8 marks)**

Determine the following:

(a) $\int 12e^{4x-1}\,dx$. (1 mark)

Solution
$3e^{4x-1} + c$
Specific behaviours
✓ correct antiderivative, with constant of integration

(b) $\int_0^{\frac{\pi}{4}} \cos(2x)\,dx$. (2 marks)

Solution
$\left[\frac{1}{2}\sin(2x)\right]_0^{\pi/4} = \frac{1}{2} - 0 = \frac{1}{2}$
Specific behaviours
✓ correct antiderivative ✓ correct value

(c) $f'\left(\frac{\pi}{2}\right)$ when $f(x) = \dfrac{\sin(3x)}{3 + \cos(x)}$. (3 marks)

Solution
$f'(x) = \dfrac{3\cos(3x)\,(3+\cos(x)) - \sin(3x)\,(-\sin(x))}{(3+\cos(x))^2}$ $f'\left(\frac{\pi}{2}\right) = \dfrac{0 - (-1)(-1)}{(3+0)^2}$ $= -\dfrac{1}{9}$
Specific behaviours
✓ correctly uses quotient rule ✓ correctly differentiates all trig terms ✓ correctly evaluates

(d) $\dfrac{d}{dx}\int_2^x \sin(t-2)\,dt$. (1 mark)

Solution
$\sin(x-2)$
Specific behaviours
✓ correct result

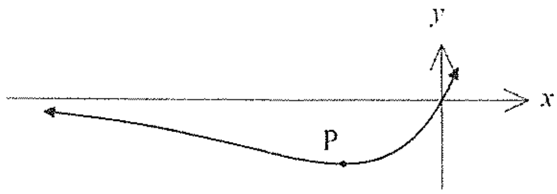
(e) $\int_0^1 \dfrac{d}{dx}(3xe^{3x})\,dx$. (1 mark)

Solution
$[3xe^{3x}]_0^1 = 3e^3$
Specific behaviours
✓ correct result

Question 2 **(7 marks)**

Let $f(x) = 2xe^{(0.5x+3)}$.

The graph of $y = f(x)$ is shown. It has one stationary point, at P, and one point of inflection.



(a) Clearly show that $f'(x) = (x+2)e^{(0.5x+3)}$. (2 marks)

Solution
$f'(x) = (2)(e^{(0.5x+3)}) + (2x)(0.5e^{(0.5x+3)})$ $= (2 + 2x \times 0.5)e^{(0.5x+3)}$ $= (x+2)e^{(0.5x+3)}$
Specific behaviours
✓ correctly differentiates exponential term ✓ shows correct use of product rule

(b) Determine the coordinates of point P. (2 marks)

Solution
$f'(x) = 0$ when $x + 2 = 0 \rightarrow x = -2$, and $f(-2) = -4e^2$. $\therefore P(-2, -4e^2)$
Specific behaviours
✓ solves $f'(x) = 0$ ✓ correctly states coordinates

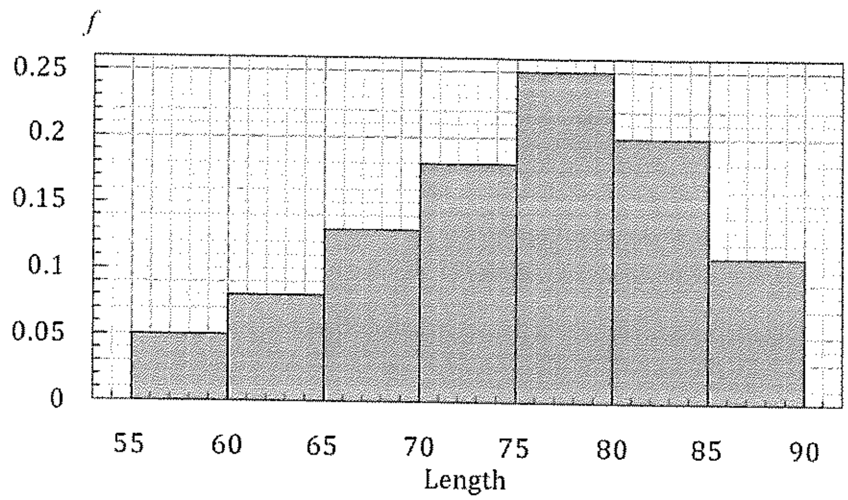
(c) Determine the values of x for which the curve $y = f(x)$ is concave down. (3 marks)

Solution
$f''(x) = (1)(e^{(0.5x+3)}) + (x+2)(0.5e^{(0.5x+3)})$ $= (0.5x+2)e^{(0.5x+3)}$ $f''(x) = 0$ when $0.5x + 2 = 0 \rightarrow x = -4$. From the graph, the curve is concave down to the left of the point of inflection and so the values of x are $x < -4$.
Specific behaviours
✓ correctly obtains $f''(x)$ ✓ indicates x-coordinate of point of inflection ✓ correct inequality for x

$f''(x) > 0$ concave up
$f''(x) < 0$ concave down.

Question 3 **(6 marks)**

The relative frequency histogram below shows the distribution of the lengths in centimetres of a large sample of fish bred in an offshore fish farm.



(a) Use the distribution to determine the probability that

(i) a randomly selected fish will be shorter than 85 cm. (1 mark)

Solution
$P(X < 85) = 1 - 0.11 = 0.89$
Specific behaviours
✓ correct probability

(ii) a randomly selected fish will be exactly 62 cm long. (1 mark)

Solution
$P(X = 62) = 0$
Specific behaviours
✓ correct probability

(iii) when two fish are randomly selected, one is shorter than 60 cm and the other is not. (2 marks)

Solution
$p = 0.05 \times 0.95 \times 2 = 0.095$ OR $\frac{1}{20} \times \frac{19}{20} \times 2 = \frac{19}{200}$ 0.095
Specific behaviours
✓ correct probabilities for each fish ✓ correct probability

(b) An observer claimed that the distribution of the lengths of fish was approximately normal with a mean of 66 cm and standard deviation of 15 cm. Comment on this claim. (2 marks)

Solution
Example comments: left skewed. - distribution not normal as histogram not bell-shaped / has negative skew, etc. - claimed mean of 66 cm is too low, should be higher, etc. - claimed sd of 15 cm is too high, should be lower, etc.
Specific behaviours
✓ one reasonable comment that refers to claim ✓ second reasonable comment that refers to claim

Question 4 **(8 marks)**

(a) The velocity, v cm per second, of electrically powered model car A at time t seconds is given by $v = \sqrt{4t+2}$. Determine the change in displacement of this car between $t = 0.5$ and $t = 3.5$ seconds. (4 marks)

Solution
$$\Delta x = \int_{0.5}^{3.5} (4t+2)^{\frac{1}{2}}\,dt$$ $$= \left[\frac{2}{3\times 4}(4t+2)^{\frac{3}{2}}\right]_{0.5}^{3.5}$$ $$= \left[\frac{1}{6}(16)^{\frac{3}{2}}\right] - \left[\frac{1}{6}(4)^{\frac{3}{2}}\right]$$ $$= \frac{1}{6}(64-8)$$ $$= \frac{28}{3}\ \text{cm/s}$$ $(9.\dot{3}\ cm/s)$
Specific behaviours
✓ writes integral for change in displacement ✓ obtains antiderivative ✓ substitutes upper and lower bounds and starts simplification ✓ correct change in displacement

$\left[\frac{(4t+2)^{\frac{3}{2}}}{6}\right]_{0.5}^{3.5}$

(b) The speed, s cm per second, of model car B at time t seconds is given by $s = e^{\sqrt{4t+2}}$, so that when $t = 3.5$, its speed was 54.6 cm per second. Use the increments formula to determine a decimal approximation for the speed of this car when $t = 3.6$. (4 marks)

Solution
Let $t = 3.5, \delta t = 3.6 - 3.5 = 0.1$ and $s = e^u$ where $u = \sqrt{4t+2}$. Then $$\frac{du}{dt} = \frac{4}{2\sqrt{4t+2}}$$ Hence $$\frac{ds}{dt} = \frac{2}{\sqrt{4t+2}}e^{\sqrt{4t+2}}\Big\vert_{t=3.5}$$ $$= \frac{2}{4}(54.6) = 27.3$$ Using increments formula $$\delta s \approx \frac{ds}{dt}\delta t$$ $$\approx 27.3 \times 0.1$$ $$\approx 2.73$$ Hence approximate speed of car is 54.6 + 2.73 = 57.33 cm/s.
Specific behaviours
✓ indicates correct derivative for u wrt to t ✓ indicates correct derivative for s wrt to t ✓ shows correct use of increments formula ✓ obtains speed of car

$\frac{ds}{dt} = e^{(4t+2)^{\frac{1}{2}}} \cdot \frac{1}{2}(4t+2)^{-\frac{1}{2}} \cdot 4$

$= \frac{2e^{\sqrt{4t+2}}}{\sqrt{4t+2}}$

$\frac{ds}{dt} = \frac{2}{\sqrt{4(3.5)+2}} \cdot 54.6$

$= \frac{54.6}{2}$

$= 27.3$

Question 5 **(7 marks)**

A computer program scans selected text messages passing through a network to see if the message contains a particular keyword. The random variable X takes the value 0 if the keyword is not found, the value 1 if it is found, and has probability distribution

$$P(X=x)=\begin{cases}\dfrac{e^{kx}}{3} & x=0,1\\ 0 & \text{elsewhere.}\end{cases}$$

(a) Show that the value of the constant k is $\log_e(2)$. (2 marks)

Solution
$P(x=0)+P(x=1)=1 \rightarrow \dfrac{1}{3}+\dfrac{e^k}{3}=1$ $e^k=2 \Rightarrow k=\log_e(2)$
Specific behaviours
✓ correctly substitutes $x=0$ and $x=1$ ✓ uses sum of probabilities to form equation and derive value of k

(b) Determine the mean and standard deviation of X. (2 marks)

Solution
$\mu=P(X=1)=\dfrac{2}{3}$ $\sigma=\sqrt{p(1-p)}=\sqrt{\dfrac{2}{3}\times\dfrac{1}{3}}=\dfrac{\sqrt{2}}{3}$
Specific behaviours
✓ correct mean   ✓ correct standard deviation

(c) Determine the probability that the program finds the keyword in exactly one of the next five randomly selected text messages that it scans. (3 marks)

Solution
$Y \sim B\left(5,\dfrac{2}{3}\right)$ $P(Y=1)=\binom{5}{1}\left(\dfrac{2}{3}\right)^1\left(\dfrac{1}{3}\right)^4$ $=\dfrac{5\times 2}{3^5}$ $=\dfrac{10}{243}$ $=(0\cdot 04115)$
Specific behaviours
✓ indicates correct binomial distribution   ✓ correct expression for probability   ✓ correct, simplified probability

Question 6 **(8 marks)**

Components A and B form part of an electronic circuit, and properties of these components are measured t seconds after the circuit is turned on.

(a) The rate of change of temperature, T °C, of component A is given by $\frac{dT}{dt} = \frac{16t}{4t^2+3}$. Determine, in simplest form, the increase in temperature of this component during the first 3 seconds. (4 marks)

Solution
$\Delta T = \int_0^3 \frac{16t}{4t^2+3}\,dt$ $= 2\int_0^3 \frac{8t}{4t^2+3}\,dt$ $= 2[\ln(4t^2+3)]_0^3$ $= 2(\ln(39) - \ln(3))$ $= 2\ln(13)$ °C
Specific behaviours
✓ writes integral to evaluate total change   ✓ integrates rate of change   ✓ substitutes limits of integral   ✓ correct increase, simplified (also accept $\ln(169)$)

(b) The current, I amps, flowing through component B reaches a peak very quickly and then declines as time goes on, as modelled by $I(t) = \frac{4+\ln(t)}{5t}$. Determine, in simplest form, the maximum current that flows through this component. (4 marks)

Solution
$I'(t) = \frac{\left(\frac{1}{t}\right)(5t) - (4+\ln t)(5)}{(5t)^2}$ $= \frac{5 - 5(4+\ln t)}{5 \times 5t^2}$ $= \frac{-3-\ln t}{5t^2}$ $I'(t) = 0 \Rightarrow \ln t = -3$ $t = e^{-3}$ $I(e^{-3}) = \frac{4-3}{5e^{-3}} = \frac{e^3}{5}$ Maximum current is $\frac{e^3}{5}$ amps.
Specific behaviours
✓ uses quotient rule correctly   ✓ obtains derivative   ✓ obtains root of derivative   ✓ calculates maximum current in simplified form

Question 7 **(8 marks)**

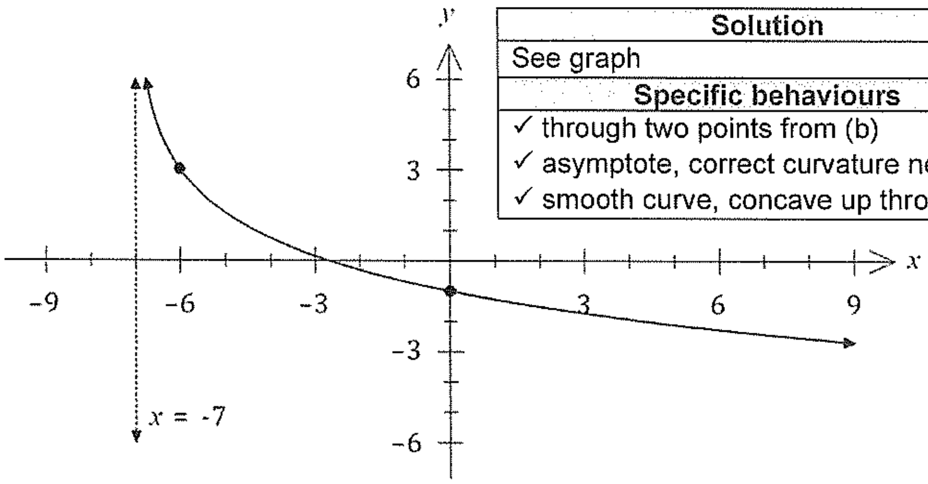
Let $f(x) = k\log_7(x+7) + c$, where k and c are constants.

The graph of $y = f(x)$ intersects line L with equation $3y + 2x + 3 = 0$ when $x = 0$ and $x = -6$.

(a) Determine the value of the constant c and the value of the constant k. (3 marks)

Solution
$x = 0 \to y = -\dfrac{0+3}{3} = -1, \qquad x = -6 \to y = -\dfrac{-12+3}{3} = 3$ Using $(-6, 3)$: $3 = k\log_7(1) + c \to c = 3$ Using $(0, -1)$: $-1 = k\log_7(7) + 3 \to k = -4$
Specific behaviours
✓ calculates two points on curve   ✓ value of c ✓ value of k

(b) Sketch the graph of $y = f(x)$ on the axes below. (3 marks)



Solution
See graph
Specific behaviours
✓ through two points from (b)   ✓ asymptote, correct curvature nearby   ✓ smooth curve, concave up throughout

(c) Given that $\log_7(x+7) = \dfrac{\ln(x+7)}{\ln(7)}$, determine the value of x where the slopes of $y = f(x)$ and line L are the same. (2 marks)

Solution
$f(x) = -4\log_7(x+7) + 3 = \dfrac{-4}{\ln(7)}\ln(x+7) + 3$ $f'(x) = \dfrac{-4}{\ln(7)} \times \dfrac{1}{x+7}$ $\dfrac{-4}{\ln(7)} \times \dfrac{1}{x+7} = \dfrac{-2}{3} \to x = \dfrac{6}{\ln(7)} - 7$
Specific behaviours
✓ correctly differentiates f ✓ solves for x-coordinate

End of questions

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SN042-205-4

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **twelve** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 8 **(8 marks)**

The launch speed of a small projectile fired from a catapult was measured and found to be normally distributed with a mean of 15.8 ms^{-1} and a standard deviation of 0.17 ms^{-1}.

(a) Determine the probability that the projectile is launched with a speed exceeding 16 ms^{-1}. (1 mark)

Solution
$P(X > 16) = 0.1197$
Specific behaviours
✓ correct probability

(b) Determine the probability that the projectile is launched with a speed exceeding 15.7 ms^{-1} given that its launch speed is less than 16 ms^{-1}. (2 marks)

Solution
$P(X > 15.7 \mid X < 16) = \dfrac{P(15.7 < X < 16)}{P(X < 16)} = \dfrac{0.6021}{1 - 0.1197} = 0.6840$
Specific behaviours
✓ indicates both probabilities required ✓ correct probability

(c) In a series of 20 launches, determine the probability that the speed of the projectile exceeds 16 ms^{-1} in no more than 3 of these launches. (2 marks)

Solution
$Y \sim B(20, 0.1197)$, $P(Y \le 3) = 0.7886$
Specific behaviours
✓ indicates binomial distribution with parameters ✓ correct probability

(d) The projectile is expected to have a speed exceeding v ms once in every 200 launches. Determine the value of v. (1 mark)

Solution
$P(X > v) = 0.005$, $v = 16.238 \text{ ms}^{-1}$.
Specific behaviours
✓ correct speed (at least 2 dp)

(e) The instrument used to measure the launch speed was suspected to overestimate the speed of the projectile by 0.03 ms^{-1}. If this was the case, state the true mean and standard deviation of the distribution of launch speeds for the projectile. (2 marks)

Solution
Mean: $\mu = 15.8 - 0.03 = 15.77 \text{ ms}^{-1}$. SD unchanged: $\sigma = 0.17 \text{ ms}^{-1}$.
Specific behaviours
✓ correct mean ✓ correct sd

Question 9 **(7 marks)**

Naltrexone is useful in managing heroin-dependent patients who find it difficult to shift away from dependent use patterns. The blood naltrexone level N of a patient who has received a naltrexone implant was observed to halve every 30 days, from an initial level of 6.9 ng/ml. The level can be modelled by an equation of the form $N = ae^{kt}$, where t is the time in days since the implant was received.

(a) State the value of the constant a and the determine the value of the constant k. (3 marks)

Solution
$a = 6.9$ $0.5 = e^{30k} \Rightarrow k = -0.0231$
Specific behaviours
✓ value of a ✓ writes equation using half-life ✓ value of k

The treatment is effective whilst the naltrexone level remains above 1.5 ng/ml.

(b) Determine the number of days that the implant will be effective. (2 marks)

Solution
$1.5 = 6.9e^{-0.0231t}$ $t = 66.05$ Implant effective for 66 days.
Specific behaviours
✓ writes equation ✓ number of days

(c) Determine the rate at which the naltrexone level is decreasing 5 weeks after the implant is received. (2 marks)

Solution
$\frac{dN}{dt} = -0.0231(6.9e^{-0.0231t})\|_{t=35}$ $= -0.071$ Hence decreasing at 0.071 ng/ml/day.
Specific behaviours
✓ indicates correct method ✓ correct rate of decrease

Question 10 **(8 marks)**

(a) A polynomial function is defined by $f(x) = (kx-1)^3$, where k is a constant. The area under the curve $y = f(x)$ between $x = 1$ and $x = 3$ is 78 square units.

Determine the area under the curve $y = f(x)$ between $x = 1$ and $x = 2$. (4 marks)

Solution
$\int_1^3 (kx-1)^3\,dx = \left[\frac{1}{4k}(kx-1)^4\right]_1^3 = \frac{(3k-1)^4-(k-1)^4}{4k}$ But $\frac{(3k-1)^4-(k-1)^4}{4k} = 78$ $k = 2$ Hence $\int_1^2 f(x)\,dx = 10 \text{ sq units}$
Specific behaviours
✓ integral for area under curve ✓ forms equation in k using given area ✓ value of k ✓ correct area

(b) The graph of another polynomial $y = g(x)$ has a point of inflection at $(2, -21)$ and a stationary point when $x = 5$.

If $g'(x) = 3x^2 + ax + b$, where a and b are constants, determine $g(x)$. (4 marks)

Solution
Since $g''(2) = 0$ then $6(2) + a = 0 \Rightarrow a = -12$. Since $g'(5) = 0$ then $3(5)^2 - 12(5) + b = 0 \Rightarrow b = -15$. $g(x) = \int 3x^2 - 12x - 15\,dx$ $= x^3 - 6x^2 - 15x + c$ Since $f(2) = -21$ then $8 - 24 - 60 + c = -21 \Rightarrow c = 25$. Hence $g(x) = x^3 - 6x^2 - 15x + 25$.
Specific behaviours
✓ value of a ✓ value of b ✓ antiderivative of $g'(x)$ ✓ evaluates constant of integration and states $g(x)$

Question 11 **(7 marks)**

The owners of a shopping mall wanted to confirm their estimate that 35% of local school students visited their mall at least once a week. The owners considered the following three ways of selecting a sample:

A Ask students who turn up to the mall after school.

B Create an online survey and publish a link to it in the local newspaper.

C Visit local homes chosen at random and ask students who live there.

(a) Briefly discuss a source of bias in each sampling method and suggest a better sampling procedure. (4 marks)

Solution
A: Non-response, students might not want to divulge information when asked. A: Undercoverage, will not sample students who don't visit mall after school. A: Convenience, only sample students who visit mall after school. B: Undercoverage, will not sample students who don't see link in newspaper. B: Self-selection, only sample students who volunteer to take survey. C: Non-response, students might not want to divulge information when asked.
Specific behaviours
✓ discusses a source of bias in A ✓ discusses a source of bias in B ✓ discusses a source of bias in C ✓ describes procedure involving random sampling from whole population

(b) It was found that 105 out of a random sample of 375 students visited the mall at least once a week. Determine the 95% confidence interval for the proportion based on this data and use it to comment on the owner's estimate. (3 marks)

Solution
$p = \frac{105}{375} = 0.28, \qquad 0.28 \pm 1.96\sqrt{\frac{0.28(1-0.28)}{375}} \approx (0.2346, 0.3254)$ The 95% confidence interval does not contain the owner's estimate of 0.35, and it suggests that the true value of the proportion is likely to be less than 35%.
Specific behaviours
✓ indicates correct method to construct confidence interval ✓ correct confidence interval (to at least 2 dp) ✓ uses interval to dispute owner's estimate

Question 12 **(10 marks)**

An online retailer of auto parts knows that on average, 18.5% of parts sold will be returned.

(a) Let the random variable X be the number of parts returned when a batch of 88 parts are sold.

(i) Describe the distribution of X. (2 marks)

Solution
X is binomially distributed with parameters $n = 88$ and $p = 0.185$. or $X \sim B(88, 0.185)$
Specific behaviours
✓ states binomial ✓ states correct parameters

(ii) Determine the probability that less than 15% of the parts sold in this batch will be returned. (2 marks)

Solution
$0.15 \times 88 = 13.2$ $P(X \leq 13) = 0.2264$
Specific behaviours
✓ indicates correct binomial probability to calculate ✓ correct probability

The retailer takes a large number of random samples of 150 parts from its sales data and records the proportion $\hat{p}$ of returned parts in each sample. Under certain circumstances, the distribution of $\hat{p}$ will approximate normality.

(b) Explain why the retailer can expect the distribution of $\hat{p}$ to closely approximate normality in this case. (3 marks)

Solution
The sampling is random (each observation is independent). The sample size is sufficiently large (typically 30 or more). At least* 15 returns and 15 non-returns can be expected in each sample. or $n\hat{p} \geq 15$ and $n(1-\hat{p}) \geq 15$. *15 *seems to be currently accepted practice, but also accept* 5 *(or more).*
Specific behaviours
✓ states samples are randomly selected ✓ states sample size sufficiently large ✓ states least number of successes and failures required

(c) State the parameters of the normal distribution that $\hat{p}$ approximates and use this distribution to determine the probability that the proportion of returns in a random sample of 150 parts is less than 15%. (3 marks)

Solution
$\hat{p} \sim N(\mu_{\hat{p}}, \sigma_{\hat{p}}^2)$ $\mu_{\hat{p}} = p = 0.185$ $\sigma_{\hat{p}} = \sqrt{\dfrac{p(1-p)}{n}} = \sqrt{\dfrac{0.185(1-0.185)}{150}} \approx 0.0317, \quad \sigma_{\hat{p}}^2 \approx 0.0010052$ Hence normally distributed with mean 0.185 and standard deviation 0.0317. $P(\hat{p} < 0.15) = 0.1348$
Specific behaviours
✓ states mean of distribution   ✓ states standard deviation or variance of distribution   ✓ correct probability

Question 13 **(8 marks)**

Brass ingots are cast by a metal recycling machine with masses of X kg, where X is a continuous random variable with cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 3 \\ ax^2 - bx & 3 \le x \le 4 \\ 1 & x > 4 \end{cases}$$

(a) Deduce from the cumulative distribution function that the values of the constants a and b are $a = 0.25$ and $b = 0.75$. (3 marks)

Solution
When $x = 3$ then $9a - 3b = 0$ and when $x = 4$ then $16a - 4b = 1$. Solving these equations simultaneously gives $a = 0.25$ and $b = 0.75$. (NB No not accept substitution as *deduction* is required, not *show*.)
Specific behaviours
✓ correctly uses lower bound to form first equation ✓ correctly uses upper bound to form second equation ✓ indicates method to solve pair of equations

(b) Determine the probability that a randomly selected ingot cast by the machine has a mass less than 3.8 kg. (1 mark)

Solution
$F(3.8) = \dfrac{19}{25} = 0.76$
Specific behaviours
✓ correct probability

(c) Determine the mean and standard deviation of the masses of ingots cast by the machine. (4 marks)

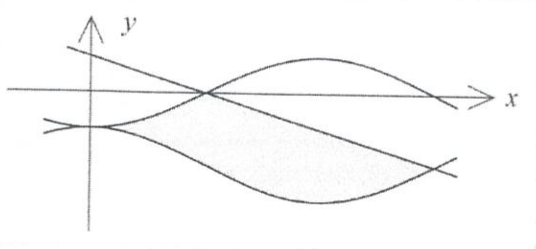
Solution
$f(x) = F'(x) = \dfrac{x}{2} - \dfrac{3}{4}$ $E(X) = \displaystyle\int_3^4 x\, f(x)\, dx = \frac{85}{24} = 3.541\bar{6}$ kg $\text{Var}(X) = \displaystyle\int_3^4 \left(x - \frac{85}{24}\right)^2 f(x)\, dx = \frac{47}{576} \approx 0.0816$ $\sigma_X = \sqrt{\text{Var}(X)} = \dfrac{\sqrt{47}}{24} \approx 0.2857$ kg
Specific behaviours
✓ obtains probability density function ✓ correct mean ✓ indicates correct integral for variance ✓ correct standard deviation

Question 14 **(7 marks)**

Functions f, g and h are defined by

$$f(x) = 10\cos\left(\frac{\pi x}{5}\right) - 20$$
$$g(x) = -10\cos\left(\frac{\pi x}{5}\right)$$
$$h(x) = 10 - 4x.$$

The graphs of these functions are shown to the right.



(a) Determine the area between $y = f(x)$, the x-axis, $x = 3.75$ and $x = 5$. (3 marks)

Solution
$I = \int_{3.75}^{5} f(x)\,dx = -\frac{25\sqrt{2}}{\pi} - 25$ $\left\| \int_{3.75}^{5} 10\cos\left(\frac{\pi x}{5}\right) - 20\,dx \right\|$ Hence area is $\frac{25\sqrt{2}}{\pi} + 25 \approx 36.3$ sq units.
Specific behaviours
✓ writes integral (may preface with negative sign - see last mark) ✓ evaluates integral ✓ clearly deals with negative value of integral to obtain area

(b) Determine the area of the shaded region enclosed by the three functions. (4 marks)

Solution
Using CAS, $f = h$ when $x = 2.5$ and $g = h$ when $x = 7.5$. $A = \int_{0}^{2.5} g(x) - f(x)\,dx + \int_{2.5}^{7.5} h(x) - f(x)\,dx$ $= \left(50 - \frac{100}{\pi}\right) + \left(50 + \frac{100}{\pi}\right)$ $= 100$ sq units
Specific behaviours
✓ writes correct integral for area between $x = 0$ and $x = 2.5$ ✓ evaluates first integral ✓ writes correct integral for area between $x = 2.5$ and $x = 7.5$ ✓ evaluates second integral and states area of shaded region

Question 15 **(9 marks)**

The number of points awarded each time an online game is played is the random variable X, where $E(X) = 3.9$ and X has the following probability distribution.

x	0	1	3	6	c
$P(X = x)$	k	0.25	0.35	0.25	0.10

(a) Determine the value of the constant c and the value of the constant k. (3 marks)

Solution
$k = 1 - (0.25 + 0.35 + 0.25 + 0.1) = 0.05$ $0 \times k + 1 \times 0.25 + 3 \times 0.35 + 6 \times 0.25 + 0.1c = 3.9$ $c = 11$
Specific behaviours
✓ value of k ✓ expression for $E(X)$ ✓ value of c

(b) Calculate the variance of Y, where $Y = 10X + 5$. (3 marks)

Solution
$\text{Var}(X) = 0.05(0 - 3.9)^2 + 0.25(1 - 3.9)^2 + 0.35(3 - 3.9)^2 + 0.25(6 - 3.9)^2 + 0.1(11 - 3.9)^2$ $= 9.29$ Hence $\text{Var}(Y) = 10^2 \times 9.29 = 929$.
Specific behaviours
✓ indicates appropriate method to determine variance or standard deviation of X ✓ variance of X ✓ variance of Y

When playing a set of 8 games, the points awarded in each game is independent of other games and a player wins a prize if the total number of points scored in the set is at least 35.

(c) A player has completed 6 games in a set and has been awarded a total of 23 points. Determine the probability that they win a prize on completion of the set. (3 marks)

Solution
Ways of getting at least 12 points from next two games: $P(11, 1 \mid 1, 11) = 2 \times 0.1 \times 0.25 = 0.05$ $P(11, 3 \mid 3, 11) = 2 \times 0.1 \times 0.35 = 0.07$ $P(11, 6 \mid 6, 11) = 2 \times 0.1 \times 0.25 = 0.05$ $P(11, 11) = 0.1^2 = 0.01$ $P(6, 6) = 0.25^2 = 0.0625$ $\sum P = 0.2425 = {}^{97}/_{400}$ Hence probability of winning a prize is 0.2425.
Specific behaviours
✓ identifies all required point combinations   ✓ correctly calculates probability of two combinations   ✓ correct probability of winning a prize

Question 16 **(9 marks)**

In a random sample of 225 adult female Australians, 72 were born overseas. This data is to be used to construct a 90% confidence interval for the proportion of adult female Australians born overseas.

(a) Determine the margin of error for the 90% confidence interval. (3 marks)

Solution
$p = 72 \div 225 = 0.32, \quad \sigma = \sqrt{\dfrac{0.32(1-0.32)}{225}} = 0.0311$ $z_{0.9} = 1.645, \quad E = 1.645 \times 0.0311 = 0.0512$
Specific behaviours
✓ correct proportion   ✓ correct standard deviation of sample proportion   ✓ correct margin of error

(b) State the 90% confidence interval. (1 mark)

Solution
$p \pm E \rightarrow (0.2688, 0.3712)$
Specific behaviours
✓ correct interval

(c) If 8 similar samples are taken and each used to construct a 90% confidence interval, determine the probability that no more than 6 of the intervals will contain the true proportion of adult female Australians who were born overseas. (2 marks)

Solution
$X \sim B(8, 0.9)$ $P(X \leq 6) = 0.1869$
Specific behaviours
✓ indicates binomial distribution with parameters   ✓ correct probability

(d) The 90% confidence interval for the proportion of adult male Australians born overseas constructed from another random sample was $(0.288, 0.412)$. Determine the number of adult males who were born overseas in this sample. (3 marks)

Solution
$E = (0.412 - 0.288) \div 2 = 0.062, \quad p = 0.288 + 0.062 = 0.35$ $\sqrt{\dfrac{0.35(1-0.35)}{n}} = \dfrac{0.062}{1.645} \rightarrow n = 160$ $X = 160 \times 0.35 = 56$ males.
Specific behaviours
✓ calculates p and E ✓ calculates sample size n ✓ correct number of females

$n = \left(\frac{k}{ME}\right)^2 \cdot \hat{p}(1-\hat{p})$

Question 17 **(10 marks)**

The turbidity index I (a measure of purity) of water being treated in tank A can be modelled by the relationship $I = 8e^{-0.2t}$, where t is the time in hours since treatment began.

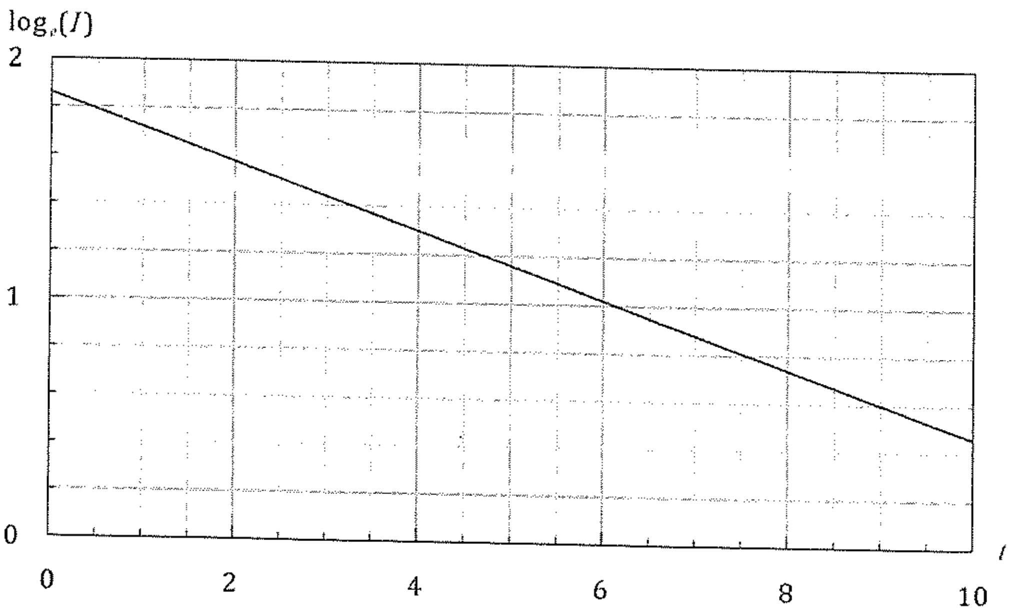
(a) Express this relationship in the form $t = p\log_e(kI)$, where p and k are constants. (2 marks)

Solution
$\frac{1}{8}I = e^{-0.2t}$ $\log_e\left(\frac{1}{8}I\right) = -0.2t$ $t = -5\log_e\left(\frac{1}{8}I\right) \quad \left[= -5\log_e\left(\frac{I}{8}\right) = -5\log_e(0.125I)\right]$ $k = \frac{1}{8}, p = -5$
Specific behaviours
✓ correctly converts from exponential to natural log form ✓ simplifies into required form

(b) Determine the time taken, to the nearest minute, for the turbidity index of the water in tank A to halve. (2 marks)

Solution
$I_0 = 8$ $t = -5\log_e\left(\frac{4}{8}\right) = 3.4657\text{h} = 3\text{h } 28\text{m}$ (208 mins)
Specific behaviours
✓ correct expression for time ✓ correct time, to nearest minute

Readings of water being treated in tank B were used to construct the graph below, where a linear relationship between $\log_e(I)$ and time t exists. The line passes through the points $(4, 1.3)$ and $(9, 0.6)$.



(c) Determine the turbidity index of the water in tank B when $t = 4$. (1 mark)

Solution
$\log_e(I) = 1.3 \to I = e^{1.3} = 3.67$
Specific behaviours
✓ correct index

(d) Determine the equation of the linear relationship shown in the graph in the form $\log_e(I) = at + b$, where a and b are constants and hence express the turbidity index I as a function of time t for the water being treated in tank B. (3 marks)

Solution

Equation of line using $y = mx + c$:

$$m = \frac{1.3 - 0.6}{4 - 9} = -\frac{7}{50} = -0.14$$
$$y - 1.3 = -0.14(x - 4)$$
$$y = -0.14x + 1.86 \to b = 1.86 = \frac{93}{50}$$

Hence

$$\log_e(I) = 1.86 - 0.14t$$

Turbidity index:

$$I(t) = e^{1.86 - 0.14t} \quad [= 6.4237e^{-0.14t}]$$

Specific behaviours

✓ calculates slope and intercept (possibly using CAS)
✓ correct equation for $\log_e(I)$
✓ correct function for I

*Use of classpad (linear regression)

Treatment began at 1:15 pm in tank A, and at 1:30 pm in tank B.

(e) Determine the time at which the turbidity indices of the water in the tanks first become the same. (2 marks)

Solution

Using $t = 0$ at 1:15 pm: $8e^{-0.2t} = e^{1.86 - 0.14(t - 0.25)} \to t = 3.074\text{h} = 3\text{h } 4\text{m}$.

or

Using $t = 0$ at 1:30 pm: $8e^{-0.2(t + 0.25)} = e^{1.86 - 0.14(t)} \to t = 2.824\text{h} = 2\text{h } 49\text{m}$.

Hence turbidity indices the same 3h 4m after 1:15 pm, at 4:19 pm.

Specific behaviours

✓ correct equation for t
✓ correct time of day

Question 18 **(8 marks)**

A small body moves along the x-axis with acceleration t seconds after leaving the origin given by $a(t) = 3.6 + kt$ cm/s², where k is a constant. The initial velocity of the body is -10 cm/s, and its change in displacement during the fifth second is 3.76 cm.

(a) Determine the maximum velocity of the body. (6 marks)

Solution
Expression for velocity $v(t) = \int 3.6 + kt\,dt$ $= 3.6t + \frac{kt^2}{2} + c$ $v(0) = -10 \Rightarrow c = -10$ Change in displacement $\Delta x = \int_4^5 3.6t + \frac{kt^2}{2} - 10\,dt$ $= \left[1.8t^2 + \frac{kt^3}{6} - 10t\right]_4^5$ $= \frac{61k}{6} + \frac{31}{5}$ Hence $\frac{61k}{6} + \frac{31}{5} = 3.76 \Rightarrow k = -\frac{6}{25} = -0.24$ Maximum velocity when no acceleration $3.6 - 0.24t = 0 \Rightarrow t = 15\text{ s}$ Maximum velocity $v(15) = 17\text{ cm/s}$
Specific behaviours
✓ obtains correct expression for velocity ✓ correct integral for change in displacement ✓ obtains linear expression for change in displacement ✓ obtains correct value of k ✓ obtains time of maximum velocity ✓ correct maximum velocity

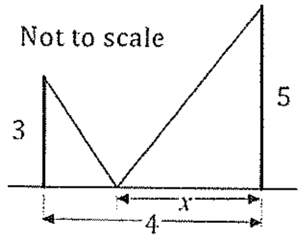
(b) Determine, to the nearest centimetre, the distance travelled by the body between $t = 0$ and the instant it reaches its maximum velocity. (2 marks)

Solution
$d = \int_0^{15} \lvert v(t) \rvert\,dt$ $\approx 149.8 \approx 150\text{ cm}$
Specific behaviours
✓ indicates correct method to determine distance travelled ✓ correct distance travelled

Question 19 **(7 marks)**

Two thin vertical posts, one 5 m and the other 3 m tall, stand 4 m apart on horizontal ground. A small stake is positioned directly between the bases of the posts at a distance of x m from the base of the taller post.

A length of thin wire runs in a straight line from the top of one post, to the stake, and then to the top of the other post.



(a) Calculate the length of the wire when the stake is positioned midway between the bases. (1 mark)

Solution
$L = \sqrt{5^2 + 2^2} + \sqrt{3^2 + 2^2} = \sqrt{29} + \sqrt{13} \approx 8.99 \text{ m}$
Specific behaviours
✓ correct length (exact or at least 2 dp)

(b) Use a calculus method to determine where the stake should be positioned to minimise the length of wire, state what this minimum length is and justify that the length is a minimum. (6 marks)

Solution

$$L = \sqrt{5^2 + x^2} + \sqrt{3^2 + (4 - x)^2}$$

$$\frac{dL}{dx} = \frac{1}{2}\frac{2x}{\sqrt{25 + x^2}} + \frac{1}{2}\frac{2(4 - x)(-1)}{\sqrt{9 + (4 - x)^2}}$$

$$= \frac{x}{\sqrt{25 + x^2}} + \frac{x - 4}{\sqrt{x^2 - 8x + 25}}$$

$$\frac{dL}{dx} = 0 \Rightarrow x = \frac{5}{2} = 2.5 \text{ m}$$

$$L(2.5) = 4\sqrt{5} \approx 8.944 \text{ m}$$

Justify minimum using sign test

$$L'(2.4) \approx -0.04, \qquad L'(2.6) \approx 0.04$$

Hence $4\sqrt{5}$ is the minimum length as the gradient changes from $-$ve to 0 to $+$ve as x increases through 2.5.

Or using second derivative

$$L''(2.5) \approx 0.38$$

Hence $4\sqrt{5}$ is the minimum length as the function is stationary and concave up when $x = 2.5$.

Specific behaviours

✓ expression for length
✓ writes first derivative (in any form)
✓ equates first derivative to 0 and obtains solution for x
✓ states minimum length (exact or at least 2 dp)
✓ indicates use of second derivative / sign test
✓ justifies length minimum

End of questions